

Universal Bounds for $SU(3)$ Low Energy Constants

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In this paper bounds for L_1 , L_2 , and L_3 are obtained in chiral perturbation theory with three flavours. At the same time we test the compatibility of this theory with axiomatic principles. Following a recent paper we use dispersion relations to write positivity conditions that translate into bounds for the chiral low energy constants. As a first approach we consider the exact $SU(3)_V$ limit and notice that if a common mass of the order of that of the kaon is adopted for the octet of pseudo-Goldstone bosons the bounds have very large $\mathcal{O}(p^6)$ corrections. Once the positivity conditions are adapted to account for different masses, we correct the previous bounds for a physical kaon mass and find that they tighten. We observe an overlap between the experimentally determined region and the first principles forbidden region, in the space of parameters.

I. INTRODUCTION

The pioneering idea of describing the dynamics of pions at very low energies with an effective field theory was developed in Refs. [1, 2] (see also Ref. [3]) and later generalized to include the K and η particles (that is, including the s quark in the light sector) in Ref. [4]. This theory is known as chiral perturbation theory (χ PT) and its Lagrangian is organized as an infinite tower of increasing dimension operators. Beyond the lowest order an increasing number unknown of low energy constants (LECs for short) must be included. The grow of LECs is even more dramatic in the theory with three flavours [$SU(3)$] because the Cayley-Hamilton relations are less restrictive than for the $SU(2)$ theory.

In a recent paper [5] axiomatic principles such as analyticity, unitarity, and crossing symmetry were used to derive universal bounds for two $SU(2)$ chiral LECs. In Ref. [6] χ PT was confronted with axiomatic principles for the first time, and the method was generalized in Ref. [7]. Some of those bounds found in Ref. [5] were already known [8, 9], but the most stringent conditions can only be found with the procedure of Ref. [5]. It was also pointed out that the linear sigma model for $m_\sigma \lesssim 24$ m has a poor convergence when the σ field is integrated out of the action, and at least corrections up to $\mathcal{O}(m_\sigma^{-6})$ must be kept to comply with the positivity bounds.

It is the purpose of the present work to generalize those results to the $SU(3)$ theory, and in particular to extend the method to cover the situation of different masses [this is, considering $SU(3)_V$ symmetry breaking]. In this way we will find out if for three flavours χ PT suffers the same anomaly as the linear sigma model.

To our knowledge the first attempt to confront dispersion relations with three-flavour χ PT to bound linear combinations of LECs was Ref. [10]. However, in this early work, the contribution from chiral logarithms in the $\mathcal{O}(p^4)$ amplitude was ignored. This simplification becomes exact in the limit of an infinite number of colours, but for a numerical analysis better results are obtained maintaining also chiral loops. In Ref. [10] it was only possible to assert that certain linear combinations of LECs

were positive and no information about the scale at which these LECs were evaluated could be obtained.

In Ref. [11] QCD inequalities on Green functions of quark bilinear currents were used for deriving bounds on some χ PT LECs. As already pointed out in Ref. [5], we are insensitive to LECs involving external currents, and so our results do not overlap.

Since χ PT consists of an expansion in both the external momenta and quark masses, the coefficients of the expansion (that is, the LECs) cannot depend on either of them. This means that LECs do not depend on the pseudo-Goldstone bosons masses. In other words the value of chiral LECs in our universe with $m_s \neq m_u = m_d$ (we will consider the isospin limit $m_u = m_d$ throughout this paper) is the same as in “another” universe in which the $SU(3)_V$ symmetry is unbroken, $m_s = m_u = m_d$. It is common lore in the literature, for instance, to consider massless quarks for estimating the values of some LECs, but this limit is not suitable for a dispersion relation analysis. The most straightforward generalization of the method used in Ref. [5] is thus to consider the exact $SU(3)_V$ limit in which there are only five independent amplitudes.

The bounds derived in this limit have two drawbacks: first, it is not clear what common mass should be adopted for the degenerate octet, what is essential to compare our bounds with the values obtained by fitting the experimental data (usually displayed at the $\mu = m_\rho$ scale); second, the results are not very challenging. In order to assess these two problems we will repeat our analysis with the physical values for the K and η masses. In this case the dispersive integrals will imply positivity conditions only under more severe conditions. Once these are addressed the new bounds turn out to be much more restrictive, and remarkably the central values of the fitted LEC values lie precisely on the border dictated by axiomatic principles.

The paper is organized as follows: in Sec. II we derive the positivity conditions for the amplitudes corresponding to the scattering of pions, kaons and etas in the $SU(3)_V$ limit and in Sec. III we transform them into bounds for chiral LECs; in Sec. IV we adapt the posi-

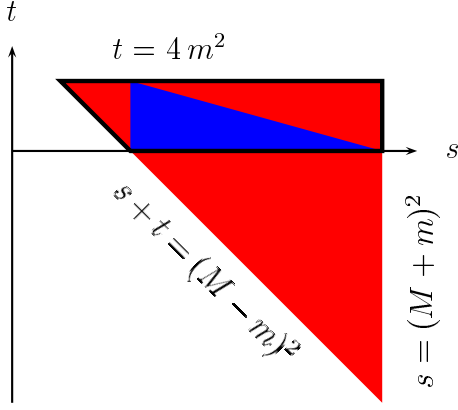


FIG. 1: Mandelstam plane for the $a + b \rightarrow a + b$ process, with $m_a = m$ and $m_b = M$ (the plot corresponds to $m = m_\pi$ and $M = m_K$). The small (blue) triangle in the center is the Mandelstam triangle. The big triangle (red and blue area) is the region free from singularities. The region bounded by the thick black line corresponds to the area \mathcal{A} in which the positivity conditions are satisfied.

tivity conditions to the situation of different masses for the pseudoscalar bosons and write a new set of positivity relations; in Sec. V we show our results; conclusions are given in Sec. VI.

II. POSITIVITY CONDITIONS IN THE $SU(3)$ LIMIT

In this section we straightforwardly apply the methods of Ref. [5] to the pseudoscalar-pseudoscalar scattering processes. As a first approach we consider the $m_u = m_d = m_s$ limit, and so the pseudoscalar octet has a common mass which we denote by m . The detailed derivation of the positivity conditions can be found in Ref [5] and will be only sketched here. Further details will be given in Sec. IV when we consider flavour symmetry breaking. Since there is no lighter particle in the QCD spectrum than the pseudo-Goldstone bosons (pGs for short) the analytic structure is fully dictated by two-pGs intermediate states. Much as happened in $\pi\pi$ scattering, the branch cuts emerge for $s, t, u > 4m^2$ what is equivalent to say that the amplitude is analytic in the Mandelstam plane for $s, t \leq 4m^2$ and $s+t \geq 0$. This result relies on perturbation theory to all orders [12], but using solely axiomatic principles it can be shown (Ref. [13]) that they are at least valid in the interval $-14m^2 \geq t \geq 4m^2$, which is enough for our purposes.

In the limit we are considering the QCD Lagrangian exhibits an exact $SU(3)_V$ symmetry. Then particles are classified according to the different irreducible representations of this group (*e.g.* pGs belong to the real octet representation) and the Wigner-Eckart theorem drastically reduces the number of independent amplitudes to six. To see this we simply need to look at the

Clebsch-Gordan decomposition of the direct product of two octets:

$$8 \otimes 8 = 27 \oplus 10 \oplus 10^* \oplus 8_1 \oplus 8_2 \oplus 1. \quad (1)$$

On the other hand one can find a representation analogous to the Chew-Mandelstam in $SU(3)$ ¹

$$\begin{aligned} T(ab \rightarrow cd) = & A_1(s, t, u) \delta^{ab} \delta^{cd} + A_2(s, t, u) \delta^{ac} \delta^{bd} \\ & + A_3(s, t, u) \delta^{ad} \delta^{bc} + B_1(s, t, u) d^{abe} d^{cde} \\ & + B_2(s, t, u) d^{ace} d^{bde}. \end{aligned} \quad (2)$$

Since Eq. (2) has only five independent amplitudes there must be one identity relating the amplitudes of Eq. (1). In fact crossing symmetry forces $T_{10}(s, t) = T_{10^*}(s, t)$, making Eqs. (1) and (2) compatible. We also expect crossing symmetry to further reduce the number of independent functions. In case of having r irreducible representation amplitudes [$r = 3$ for $SU(2)$ and $r = 6$ for $SU(3)$] crossing symmetry implies that there are only $\frac{r}{3}$ independent functions. This is easy to understand: the r irreducible functions T^I $I = 1, \dots, r$ translate into $3r$ degrees of freedom $T^I(s, t)$, $T^I(t, s)$, and $T^I(4m^2 - s - t, t)$ corresponding to the s -, t -, and u -crossed channels, respectively. Crossing symmetry implies $2r$ restrictions, since it relates the s -channel amplitudes with the t - and u -channel ones ($2r$ relations). So we end up with r independent degrees of freedom, which is equivalent to $\frac{r}{3}$ independent functions. So in $\pi\pi$ scattering there is only one independent function (*e.g.* the Chew-Mandelstam coordinate A) while we are left with two independent functions. All in all for $SU(3)$ we can write the following crossing relation

$$T^I(s, t) = C_u^{II'} T^{I'}(u, t), \quad C_u^{II'} C_u^{I'J} = \delta_{IJ},$$

$$C_u = \begin{pmatrix} \frac{7}{40} & \frac{1}{6} & \frac{1}{5} & \frac{1}{3} & \frac{1}{8} \\ \frac{9}{40} & \frac{1}{2} & \frac{2}{5} & 0 & -\frac{1}{8} \\ \frac{27}{40} & 1 & -\frac{3}{10} & -\frac{1}{2} & \frac{1}{8} \\ \frac{9}{8} & 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{8} \\ \frac{27}{8} & -\frac{5}{2} & 1 & -1 & \frac{1}{8} \end{pmatrix}, \quad (3)$$

and analogously for $T^I(t, s)$. We use $I, J = 27, 10, 8_1, 8_2, 1$ to denote the irreducible amplitudes of Eq. (1), not to be confused with isospin. Exchanging the order of the initial or final particle amounts to change t by $4m^2 - s - t$. Under this operation the amplitudes $I = 1, 8_1, 27$ remain invariant and the rest change sign.

¹ One must remember the $SU(3)$ identity $3(d^{abe} d^{cde} + d^{ace} d^{bde} + d^{ade} d^{bce}) = \delta^{ab} \delta^{cd} + \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc}$ to make sure that the basis of tensors is minimal. One can also add four more structures of the type $f^{abe} d^{cde}$, but they clash after imposing crossing symmetry.

Following Ref. [5] we can write the following twice-subtracted dispersion relation

$$\frac{d^2}{ds^2} T^I(s, t) = \frac{2}{\pi} \int_{4m^2}^{\infty} dx \left[\frac{\delta^{II'}}{(x-s)^3} + \frac{C_u^{II'}}{(x-u)^3} \right] \text{Im} T^{I'}(x + i\epsilon, t), \quad (4)$$

wherever (s, t) makes the amplitude analytic, that is $t \leq 4m^2$, $s+t \geq 0$ and if $s > 4m^2$ considering $s \rightarrow s+i\epsilon$, corresponding to the Feynman prescription for propagators. Clearly, if we restrict ourselves to $s < 4m^2$ and

$s+t > 0$, both denominators in Eq. (4) are positive. As shown in Ref. [5], for several linear combinations $\sum_I a_I T^I$ with $a_I \geq 0$, $\sum_I a_I C_u^{IJ} T_J \equiv \sum_J b_J T_J$ with $b_J = \sum_I a_I C_u^{IJ} \geq 0$. These have a positive imaginary part along the integral for $t > 0$, corresponding to physical processes with equal initial and final states. Of course, many different processes are related by $SU(3)$ symmetry and need to be considered only once. If a process can be expressed as a linear combination of other processes with positive coefficients it cannot be more restrictive than the processes separately, so it will be discarded. With all that we obtain the following set of positivity conditions:

$$\begin{aligned} \frac{d^2}{ds^2} T(\pi^+\pi^+ \rightarrow \pi^+\pi^+)[(s, t) \in \mathcal{A}] &\geq 0, \\ \frac{d^2}{ds^2} T(\pi^+\pi^0 \rightarrow \pi^+\pi^0)[(s, t) \in \mathcal{A}] &\geq 0, \\ \frac{d^2}{ds^2} T(K\eta \rightarrow K\eta)[(s, t) \in \mathcal{A}] &\geq 0, \\ \frac{d^2}{ds^2} T(\pi^0\pi^0 \rightarrow \pi^0\pi^0)[(s, t) \in \mathcal{A}] &\geq 0, \\ \frac{d^2}{ds^2} T(\eta\pi \rightarrow \eta\pi)[(s, t) \in \mathcal{A}] &\geq 0, \\ \frac{d^2}{ds^2} T(K\pi^+ \rightarrow K\pi^+)[(s, t) \in \mathcal{A}] &\geq 0, \end{aligned} \quad (5)$$

where \mathcal{A} is the closed region of the Mandelstam plane defined by $0 \leq t \leq 4m^2$, $s \leq 4m^2$, $s+t \geq 0$ (see Fig. 1). Equation (5) corresponds to the following linear combinations of irreducible amplitudes

$$\begin{aligned} \frac{27}{20} T_{27} + \frac{1}{5} T_{8_1} + \frac{1}{8} T_1, \quad \frac{1}{2} T_{27} + \frac{1}{3} T_{8_2} + \frac{1}{5} T_{10}, \\ \frac{3}{10} T_{27} + \frac{1}{5} T_{8_1} + \frac{1}{2} T_{10}, \quad \frac{1}{2} T_{27} + \frac{1}{2} T_{10}, \\ T_{27}, \quad \frac{9}{20} T_{27} + \frac{1}{20} T_{8_1} + \frac{1}{4} T_{8_2} + \frac{1}{4} T_{10}, \end{aligned} \quad (6)$$

respectively.

III. BOUNDS ON L_1 , L_2 AND L_3 .

It is straightforward now to convert the positivity conditions in Eq. (5) into bounds for chiral LECs, since the energy domain \mathcal{A} is well inside the convergence radius of χ PT. We simply plug into Eq. (5) the $\mathcal{O}(p^4)$ χ PT prediction [the $\mathcal{O}(p^2)$ prediction vanishes when acting with two derivatives] for the different amplitudes and seek the most stringent point in \mathcal{A} . These amplitudes can be found in the literature but are collected and very nicely displayed in Ref. [14], which we follow. Upon the second derivative they only depend on three LECs: L_1 , L_2 , and L_3 . At one loop the amplitudes explicitly depend on the chiral renormalization scale μ , but it is in fact canceled by the implicit μ dependence of the chiral LECs. We will adopt the value $\mu = m$ that greatly simplifies the expressions (as it is the only energy scale in the process). So we will get our bounds for L_1 and L_2 evaluated at that

energy scale (L_3 does not get renormalized and thus it is μ independent). Our bounds have the following general expression

$$\alpha_{1i} L_1^r(m) + \alpha_{2i} L_2^r(m) + \alpha_{3i} L_3^r \geq f_i[(s, t) \in \mathcal{A}] \Big|_{\max}, \quad (7)$$

where f_i are functions obtained by isolating the LECs of the second derivative of the amplitude: it contains chiral logarithms and constant LEC-independent terms. For all processes the maximum is achieved for $t = 4m^2$. For the processes $\pi^+\pi^+ \rightarrow \pi^+\pi^+$ and $K\pi^+ \rightarrow K\pi^+$ the minima are found for $s = 1.3684m^2$ and $s = 1.2593m^2$, respectively. For the rest of the processes it is found for $s = 0$.

If we are to compare our theoretical bounds with the fitted values we need to fix the common mass m to a physical value. The most conservative value is of course the pion mass m_π , since it is the lightest particle in the octet, but in principle any value low enough not to compromise the chiral expansion is equally good. We will adopt the two extreme values m_π and m_K for our analysis. The results are shown in Table I.

If we consider the more realistic case of $m_s \neq m_u = m_d$ and use the physical value for the π and K states² the choice of m is absolutely transparent. This is discussed in the next section.

² In our analysis we will assume the Gell-Mann-Okubo formula for the masses: $m_\eta^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2$.

IV. SYMMETRY CORRECTIONS TO THE BOUNDS

The first effect showing up when considering $m_\pi < m_K$ is that for several processes the unitarity branch cut might occur before reaching the physical threshold. This, as we discuss next, spoils the positivity condition.

Let us first obtain the analytic triangle for the present situation. We will consider only processes with equal initial and final states $a+b \rightarrow a+b$, of masses $m_a = M$ and $m_b = m$ ($M \geq m$), since this ensures that the imaginary part of the partial wave amplitudes is positive. If the lowest mass intermediate state in that process is $c+d$, the amplitude is analytic for $s \leq (m_c + m_d)^2$. Analogously from the crossed channels we will obtain $t \leq (m_e + m_f)^2$ and $s+t \geq 2(m^2 + M^2) - (m_g + m_h)^2$. Of course the maximum [minimum] value for these three thresholds are $(m+M)^2$, $4m^2$, and $(M-m)^2$, respectively. Then the dispersion relation reads (now we directly consider physical processes)

$$\begin{aligned} \frac{d^2}{ds^2} T(s, t) &= \frac{2}{\pi} \int_{(m_c+m_d)^2}^{\infty} dx \frac{\text{Im } T(x + i\epsilon, t)}{(x-s)^3} \\ &+ \frac{2}{\pi} \int_{(m_g+m_h)^2}^{\infty} dx \frac{\text{Im } T_u(x + i\epsilon, t)}{(x-u)^3}, \quad (8) \end{aligned}$$

wherever the amplitude is analytic. Using only axiomatic principles [13] it can be shown that for $K\pi$ and $\eta\pi$ scattering, dispersion relations are valid at least for $-32.76m^2 \geq t \geq 4m^2$ and $-37.85m^2 \geq t \geq 4m^2$, respectively. Here T_u is the amplitude corresponding to the u -channel $a+\bar{b} \rightarrow a+\bar{b}$, which has, of course, equal initial and final states, too. Both denominators are positive as far as $s \leq (m_c + m_d)^2$ and $s+t \geq 2(m^2 + M^2) - (m_g + m_h)^2$, and so up to this point there is nothing compromising the positivity condition. But still

we have to make sure that the imaginary part remains positive along the two cuts. Expanding the amplitude T (and also T_u) in partial waves we get

$$T(s, t) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(s) P_\ell \left[1 + \frac{st}{(s+m^2-M^2)^2 - 4m^2s} \right], \quad (9)$$

with $\text{Im } f_\ell(s) = s \beta(s) \sigma_\ell(s) \theta[s - (m_c + m_d)^2] \geq 0$ and with $\theta[s - (m_g + m_h)^2]$ for the u -channel. So for getting a positive imaginary part each P_ℓ must be positive along the corresponding cuts. Since $P_\ell(z) > 1$ for $z > 1$ for all ℓ it is enough to require

$$\frac{st}{(s+m^2-M^2)^2 - 4m^2s} \geq 0 \quad \text{for } s \geq \begin{cases} (m_c + m_d)^2 \\ (m_g + m_h)^2 \end{cases}. \quad (10)$$

Since for $s \rightarrow \infty$ Eq. (10) tends to t/s then we must require $t > 0$. Then for positive t Eq. (10) is only satisfied if $(M-m)^2 \geq s \geq (M+m)^2$. Thus if either $(m_c + m_d)$ [or $(m_g + m_h)$] is less than $(M+m)$ the imaginary part between $(m_c + m_d)$ [or $(m_g + m_h)$] and the physical threshold could turn negative, making the positivity condition invalid.

Summarising, *the positivity conditions hold for processes of the type $a+b \rightarrow a+b$ such that the lightest pair of particles that can arise off the scattering $a+b$ is precisely $a+b$, and analogously for $a+\bar{b}$* . Or in other words, for processes with equal initial and final states such that the imaginary part of the s - and u -channels starts at their physical production threshold. Moreover, the positivity condition is satisfied in the closed area of the Mandelstam plane \mathcal{A} defined by $0 \geq t \geq 4m^2$, $s \leq (M+m)^2$ and $s+t \geq (M-m)^2$ (see Fig. 1). As an additional bonus for breaking $SU(3)_V$ we have many independent amplitudes that are no longer related by symmetry. The final set of positivity conditions reads:

$$\begin{aligned} \frac{d^2}{ds^2} T(\pi^+\pi^+ \rightarrow \pi^+\pi^+)[(s, t) \in \mathcal{A}_\pi] &\geq 0, \\ \frac{d^2}{ds^2} T(\pi^+\pi^0 \rightarrow \pi^+\pi^0)[(s, t) \in \mathcal{A}_\pi] &\geq 0, \\ \frac{d^2}{ds^2} T(K\pi^+ \rightarrow K\pi^+)[(s, t) \in \mathcal{A}_K] &\geq 0, \\ \frac{d^2}{ds^2} T(\pi^0\pi^0 \rightarrow \pi^0\pi^0)[(s, t) \in \mathcal{A}_\pi] &\geq 0, \\ \frac{d^2}{ds^2} T(\eta\pi \rightarrow \eta\pi)[(s, t) \in \mathcal{A}_\eta] &\geq 0, \end{aligned} \quad (11)$$

where of course, the area \mathcal{A} depends on each specific process. There are more processes satisfying the conditions stated above, but they give a less stringent bound for the same linear combination of LECs and so we will not show them. Again all minima are found at $t = 4m^2$. For the $\pi^+\pi^0$, $\eta\pi$, and $K\pi^+$ processes the minima are achieved for $s = 1.14384m^2$, $s = 16.0027m^2$, and $s = 4.78m^2$, respectively. For the remaining two processes, it is found

at $s = 0$.

V. RESULTS

In this section we discuss the bounds obtained for the different linear combinations of chiral LECs, and compare them with the values obtained by fitting observables

Process	$10^3 \alpha_i L^i(m_\rho)$	Fit to exp.	Bound $m = m_\pi$	Bound $m = m_K$	Bound $m_\pi \neq m_\pi$
$\pi^0 \pi^0$	$2 L_1^r + 2 L_2^r + L_3$	$-0.03 (1.03) \pm 0.5$	$\geq -3.88 \pm 0.20$	$\geq 0.68 \pm 2.50$	$\geq -3.87 \pm 3.00$
$\pi^+ \pi^0$	L_2^r	$0.73 (1.59) \pm 0.12$	$\geq -1.30 \pm 0.20$	$\geq 0.22 \pm 2.50$	$\geq -1.10 \pm 3.00$
$\pi^+ \pi^+$	$2 L_1^r + 3 L_2^r + L_3$	$0.70 (2.62) \pm 0.6$	$\geq -4.88 \pm 0.20$	$\geq 1.20 \pm 2.50$	$\geq -4.29 \pm 3.00$
$K \eta$	$12 L_2^r + L_3$	$6.41 (16.17) \pm 1.5$	$\geq -15.99 \pm 0.20$	$\geq 2.24 \pm 2.50$	-
$\eta \pi$	$3 L_2^r + L_3$	$-0.16 (1.86) \pm 0.5$	$\geq -3.64 \pm 0.20$	$\geq 0.92 \pm 2.50$	$\geq -0.15 \pm 3.00$
$K^+ \pi^+$	$4 L_2^r + L_3$	$0.57 (3.45) \pm 0.6$	$\geq -4.70 \pm 0.20$	$\geq 1.38 \pm 2.50$	$\geq -14.75 \pm 3.00$

TABLE I: Experimental values for linear combinations of the LECs and their bounds. In the third column, for the values displayed in brackets we use the fitted values of the LECs when using an $\mathcal{O}(p^4)$ χ PT theoretical prediction.

to the experimental data. In Ref. [15] those values are given at the $\mu = m_\rho$ scale, so we will run our bounds to this scale to compare. The running equation for these LECs reads

$$L_i(\mu_1) - L_i(\mu_2) = -\frac{\Gamma_i}{16\pi^2} \log\left(\frac{\mu_1}{\mu_2}\right),$$

$$\Gamma_1 = \frac{3}{32}, \quad \Gamma_2 = \frac{3}{16}, \quad (12)$$

and the values at the different scales are [15]

$$\begin{aligned} L_1^r(m_\rho) &= (0.43 [0.38] \pm 0.12) \times 10^{-3}, \\ L_2^r(m_\rho) &= (0.73 [1.59] \pm 0.12) \times 10^{-3}, \\ L_3 &= (-2.35 [2.91] \pm 0.37) \times 10^{-3}. \end{aligned} \quad (13)$$

Those values were obtained from a fit to the available experimental data taking as theoretical input the $\mathcal{O}(p^6)$ χ PT prediction. Since in our analysis we are using the $\mathcal{O}(p^4)$ amplitude it is instructive to compare our bounds with the values of the LECs obtained by fitting the $\mathcal{O}(p^4)$ χ PT amplitude to the same data. Those can be found in Ref. [15] as well, and are displayed in Eq. (13) in brackets.

A very important issue is to estimate the error committed by truncating the amplitude at $\mathcal{O}(p^4)$. The $\mathcal{O}(p^6)$ amplitude is divided into three pieces: two-loop terms, that only depend on masses; one-loop terms, that depend on several $\mathcal{O}(p^4)$ LECs; and tree-level terms, that depend on $\mathcal{O}(p^6)$ LECs. For the symmetric analysis the error can be estimated as in Ref. [5], that is, adopting as an educated guess 3 times the corrections due to double chiral logarithms. When assuming $m = m_\pi$ the bounds are not very stringent and the errors are rather small; experimental values are well within the bounds. However, for $m = m_K$ the central values of the bounds greatly increase (that is, bounds tighten) and some experimental values apparently violate the bounds. But at the same time errors get multiplied by a factor of 12. Thence the validity of the chiral expansion is not compromised.

For the symmetry breaking analysis the error cannot be estimated so straightforwardly. It is expected that the main corrections come from chiral LECs multiplied by the kaon mass. The $\mathcal{O}(p^6)$ computation of the $\pi\pi$

scattering amplitude in three-flavour χ PT was performed in Ref. [16], and the $K\pi$ scattering at the same order can be found in Ref. [17]. We will adopt as an educated guess the correction due to the $\mathcal{O}(p^6)$ LECs, that is the $\mathcal{O}(p^6)$ tree-level piece. Unfortunately the $\mathcal{O}(p^6)$ LECs are unknown, so we will use the estimate given in Refs. [16, 17], obtained by resonance saturation. In addition, to be more conservative, we will assume a common error for all the channels, the biggest of these, which is 3.0. This error is very large, of the same order as that of the symmetric analysis with $m = m_K$.

For the three $\pi\pi$ scattering processes we do not see large deviations of the corrected bounds (they increase around 20%). However the estimated error due to higher order corrections greatly enhances due to terms proportional to the kaon mass. So we can conclude that the symmetric analysis is most convenient for these relations. Incidentally experimental values satisfy these three bounds. For $K\pi$ scattering the corrected bound is much worse. However for $\eta\pi$ scattering the increase of the corrected bound is great: 139%. In fact the experimentally fitted value is partially in conflict with the bound, but since the error of the bound is quite large, the validity of χ PT is not compromised.

The bounds compare better to the values of the LECs obtained from an $\mathcal{O}(p^4)$ fit. It is quite easy to understand this. The bounds are to a large extent dominated by the value of L_2 , since in the corresponding linear combinations it always appears multiplied by large coefficients (see second column of Table I). In Eq. (13) we see that the value of L_2 in the $\mathcal{O}(p^4)$ fit is twice as big as in the $\mathcal{O}(p^6)$.

Results are displayed in Table I. In the first column we show which process is rendering each bound and in the second the corresponding linear combination of LECs, in the third column we display the corresponding linear combinations of the experimentally fitted values from the $\mathcal{O}(p^6)$ fit, and in brackets when using the values from the $\mathcal{O}(p^4)$ fit; in the fourth and fifth columns we display the bounds for the symmetric analysis assuming $m = m_\pi$ and $m = m_K$, respectively; in the last column we give the bounds obtained for broken $SU(3)_V$ symmetry.

VI. CONCLUSIONS

As demonstrated in Ref. [5] the combination of effective field theories and axiomatic principles turns out to be a powerful tool for disentangling some properties of nonperturbative phenomena. The latter render model independent positivity conditions that yield bounds on the LECs of the former.

We apply this program to χ PT with three flavours and find bounds for L_1 , L_2 , and L_3 . When the exact $SU(3)_V$ limit is considered the bounds become badly convergent if the common mass m for the multiplet of pGs is of the order of m_K (albeit they converge well for $m = m_\pi$). When the actual values for the pion and kaon masses are employed the bounds become more stringent and, in fact, in one case the experimentally fitted values are

partially in contradiction with the central value of the bound. However, for this process the $\mathcal{O}(p^6)$ corrections are very large and so there is no contradiction.

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- [1] S. Weinberg, *Physica A* **96** (1979) 327.
 - [2] J. Gasser and H. Leutwyler, *Annals Phys.* **158** (1984) 142.
 - [3] S. R. Coleman, J. Wess and B. Zumino, *Phys. Rev.* **177** (1969) 2239.
 - [4] J. Gasser and H. Leutwyler, *Nucl. Phys. B* **250** (1985) 465.
 - [5] A. V. Manohar and V. Mateu, arXiv:0801.3222 [hep-ph].
 - [6] B. Ananthanarayan, D. Toublan and G. Wanders, *Phys. Rev. D* **51** (1995) 1093.
 - [7] P. Dita, *Phys. Rev. D* **59** (1999) 094007.
 - [8] M. R. Pennington and J. Portolés, *Phys. Lett. B* **344** (1995) 399.
 - [9] J. Distler, B. Grinstein, R. A. Porto and I. Z. Rothstein, *Phys. Rev. Lett.* **98** (2007) 041601.
 - [10] T. N. Pham and T. N. Truong, *Phys. Rev. D* **31** (1985) 3027.
 - [11] J. Comellas, J. I. Latorre and J. Taron, *Phys. Lett. B* **360** (1995) 109.
 - [12] R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, *The Analytic S-Matrix*, Cambridge University Press (1966).
 - [13] A. Martin, *Nuovo Cim. A* **42** (1966) 930.
 - [14] A. Gomez Nicola and J. R. Pelaez, *Phys. Rev. D* **65** (2002) 054009.
 - [15] G. Amoros, J. Bijnens and P. Talavera, *Nucl. Phys. B* **602** (2001) 87.
 - [16] J. Bijnens, P. Dhonte and P. Talavera, *JHEP* **0401** (2004) 050.
 - [17] J. Bijnens, P. Dhonte and P. Talavera, *JHEP* **0405** (2004) 036.